Computer Aided Drafting, Design and Manufacturing
Volume 25, Number 4, December 2015, Page 19

CADDM

Ambiguous Cylinders: a New Class of Impossible Objects

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Abstract: This paper presents a new class of surfaces that give two quite different appearances when they are seen from two special viewpoints. The inconsistent appearances can be perceived by simultaneously viewing them directly and in a mirror. This phenomenon is a new type of optical illusion, and we have named it the “ambiguous cylinder illusion”, because it is typically generated by cylindrical surfaces. We consider why this illusion arises, and we present a mathematical method for designing ambiguous cylinders.

Key words: ambiguous cylinder; cylindrical surface; impossible object; optical illusion

1 Introduction

An optical illusion is a phenomenon in which what we perceive is different from the physical reality. Typical classical examples include the Müller-Lyer illusion, in which the lengths of two line segments appear to be different although they are the same, and the Zöllner illusion, in which two lines appear to be nonparallel, even though they are parallel. Optical illusions are important in visual science, because they provide material for studying the human vision system[1-2].

Studies of optical illusions began with 2D pictures, but recently, they have been extended to 3D surfaces. Penrose and Penrose[3] pointed out that there are 2D pictures that give the impression of 3D surfaces that do not exist. These images are called “impossible objects”. Stimulated by their finding, many researchers have studied this class of images from an artificial intelligence point of view[4-6]. In particular, Sugihara[7-8] pointed out that “impossible objects” are not necessarily impossible, and he presented a method for constructing 3D objects from anomalous pictures. He extended his design principle to the “impossible motion illusion”, in which an object appears to be ordinary, but inserted motion appears to be physically impossible[9-10].

In this paper, we propose a new type of impossible object. That is, we have found a class of surfaces that give two quite different appearances when they are seen from two special viewpoints. An example is shown in Fig. 1, where a vertical mirror stands behind a garage, but the roof of the garage and its mirror image appear to be quite different from each other. The actual shape of the roof is different from either of these. Another example is shown in Fig. 2, in which the mirror image of a circular cylinder appears to be shaped as a star. These visual phenomena are a new type of optical illusion, which we have named the “ambiguous cylinder illusion”. In this paper, we define this class of surfaces and present a method for constructing them.

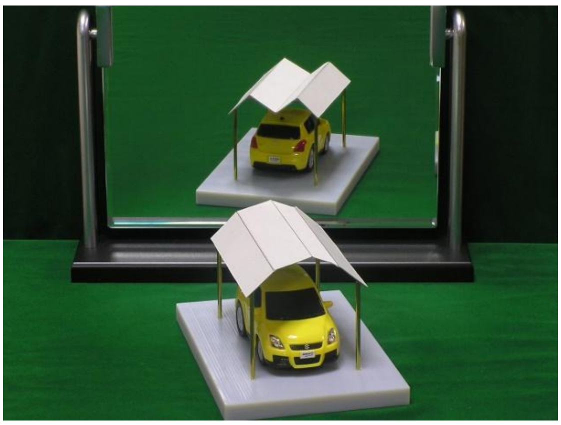

Fig. 1. Ambiguous garage roof.

2 Related Illusions

There are several types of optical illusions in which

Project Item: Supported the Grant-in-Aid for Basic Scientific Research (No.24360039) and Challenging Exploratory Research (No. 15K12067).

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a single object has two interpretations. One typical class is depth reversal; these include figure-ground reversals, Necker cubes, crater illusions, and hollow mask illusions[1-2,11]. A new illusion called Height reversal[12] is also an example belonging to this class. In this class of illusion, the shapes of the two interpretations are closely related in the sense that one interpretation is the reversal of the depth of the other. In another class of illusion, the shapes appear to be very different; these include hybrid images[13] and multiple silhouette sculptures[14]. The appearances of hybrid images depend on the distance from the viewer, and aspects of multiple silhouette sculptures are visible from mutually orthogonal viewpoints. The ambiguous cylinder illusion presented in this paper is a new class of illusion in which the surface of an object appears to be two very different shapes when viewed from two particular viewpoints.

Fig. 2. Ambiguous cylinder "A Full Moon and a Star".

3 Cylindrical Surface and Its Perception

The design of ambiguous cylinders is based on two observations, one mathematical and the other psychological. In this section, we summarize these two observations, on which, in the next section, we will establish a method for designing ambiguous cylinders.

As shown in Fig. 3, let C be a space curve, and let A be its projection onto a plane with respect to the center of the projection at E. This implies that, if we see the space curve C from the viewpoint E, the appearance of C is the same as that of A. If C is given, A is determined uniquely. On the other hand, we cannot uniquely reconstruct the space curve from a 2D picture; there are infinitely many space curves whose projections coincide with a given picture. In this sense, there is freedom in the choice of the space curve resulting from any given picture. This is summarized in the next observation.

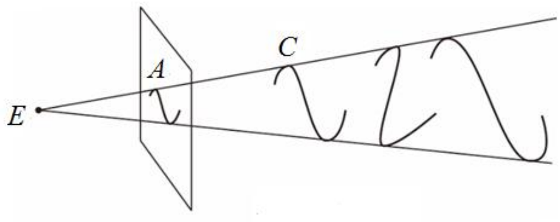


Fig. 3. Freedom in the choice of a space curve from a planar curve.

Observation 1 (ambiguity of a picture). The space curve whose projection coincides with a given planar curve is not unique. There is freedom in the choice of the space curve.

This geometrical property enables us to construct a space curve such that, for two specified viewpoints, it will generate two given planar curves.

Next suppose that, as shown in Fig. 4, we choose a line segment, say L. Suppose, too, that we move it continuously in 3D without changing its orientation and that we do this in such a way that one of its terminal points traces a given space curve C. We then form a surface from all the points which have been occupied by this line segment. We call this the *cylindrical surface* generated by the line segment L and the curve C, and we call L and C the *generating line segment* and the *guiding space curve*.

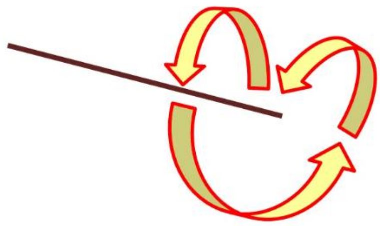


Fig. 4. Line segment with a fixed orientation sweeping through space to form a cylindrical surface.

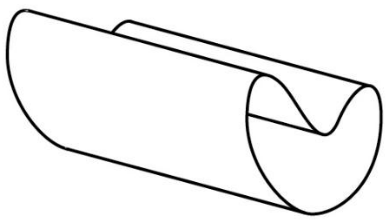


Fig. 5. Cylindrical surface generated by a line segment.

If the beginning and ending positions of the generating line segment are the same, the associated cylindrical surface is topologically equivalent to the sides of a cylinder, as shown in Fig. 5. In this case, we call the surface a *closed cylinder*. Otherwise, we call the resulting surface an *open cylinder*. The roof of the garage in Fig. 1 is an open cylinder, while the surface in Fig. 2 is a closed cylinder.

In our investigations into the human interpretation of images as 3D objects, we have found that the human brain has a strong preference for right angles[15]. In many cases, the human brain interprets an image to be the object with the maximum number of right angles, and it ignores the many other possible interpretations. This tendency is observed widely, and sometimes called "shape constancy"[2,16]. This is also true when we see cylindrical surfaces.

For example, when we view the cylindrical surface in Fig. 5, we interpret this by seeing the edge as the planar curve obtained by cutting the cylinder with a plane perpendicular to the axis of the cylinder, as shown in Fig. 6. However, there are other possibilities. As shown in Fig. 7, the edge of the cylinder can be interpreted as an intersection of the cylinder with a curved surface.

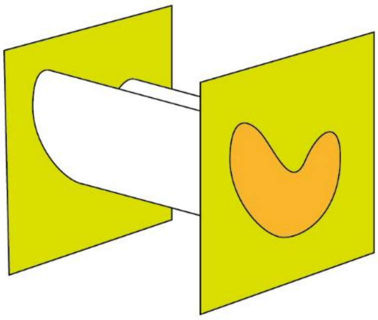


Fig. 6. Interpretation of the cylinder as a planar cut.

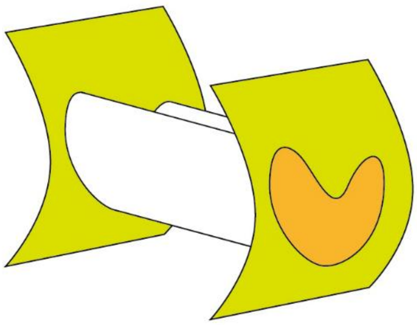


Fig. 7. Another interpretation of the cylinder.

However, the human brain usually chooses the planar curve interpretation instead of the space curve interpretation. Hence, we get the next observation.

Observation 2 (preference for the planar cut interpretation). When we see a single image of a cylindrical surface generated by a line segment L with a guiding space curve C, our brains are apt to interpret C as the planar curve obtained by cutting in a plane perpendicular to the axis L of the cylinder.

By combining these two observations, we can design a cylindrical surface that will appear to be a different chosen shape when observed from each of two special viewpoints. This is the basic idea behind designing ambiguous cylinders. We will show a detailed implementation of this idea in the next section.

4 Construction of Ambiguous Cylinders

Let $A(s)$ and $B(s)$ be two planar curves, where s is a parameter such that $0 \leqslant s \leqslant 1$. As shown in Fig. 8, we assume that the two curves are embedded in a vertical plane T, they are monotone in the horizontal direction, and the left and right terminal points coincide, that is, $A(0)=B(0)$ and $A(1)=B(1)$.

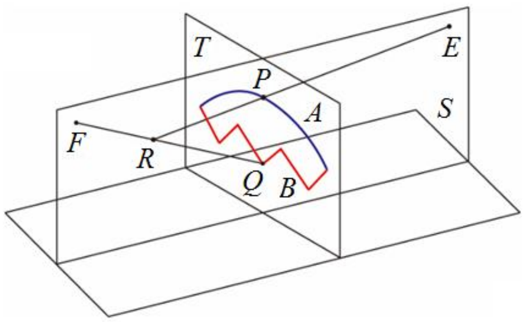


Fig. 8. Computation of a space curve that realizes two given planar curves.

Let E and F be two points that are not included in T. We want to construct a space curve $C(s)$ that appears to be $A(s)$ when it is seen from viewpoint E and that appears to be $B(s)$ when it is seen from viewpoint F. This method for achieving this goal is presented as Algorithm 1.

Algorithm 1 (construction of the required space curve)

Input: Two planar curves $A(s)$ and $B(s)$ embedded in vertical plane T, and two viewpoints E and F.

Output: Space curve $C(s)$ that coincides with $A(s)$ and $B(s)$ when seen from E and F, respectively.

Procedure:

1. Choose sufficiently many values of $s \in [0,1]$, and

for each $P=A(s)$, do the following.

1.1 Construct plane S that contains three points E, F, and P.

1.2 Find point Q, which is at the intersection of the curve B with the plane S.

1.3 Find point R on the intersection of the line EP with the line FQ, and put $C(s)=R$.

2. Report $C(s)$ and stop.

In practice, we choose a finite number of points $s_1=0< s_2<\cdots<s_{n-1}<s_n=1$, compute the values $C(s_i)$, and connect them to get a polyline $(C(s_1),C(s_2),\cdots, C(s_n))$ that approximates the space curve $C(s)$.

Note that the point Q in Step 1.2 can always be found, and it will be unique because $B(s)$ is monotone in the horizontal direction, and $A(0)=B(0)$, $A(1)=B(1)$. Furthermore, the point R in Step 1.3 can always be found, and it will be unique because the two lines EP and FQ are both on the plane S.

Algorithm 2 (construction of a cylindrical surface)

Input: Space curve $C(s)$, $0 \leqslant s \leqslant 1$, and line segment L that is perpendicular to the plane T.

Output: Cylindrical surface.

Procedure:

1. Without changing its orientation, move the line segment L in such a way that one of the terminal points of L moves along the curve $C(s)$.

2. Report the swept surface and stop.

The output surface of this algorithm is a template for constructing the ambiguous cylinder. Indeed, if we view this cylinder from E, the section coincides with $A(s)$, and if we view it from F, the section coincides with $B(s)$. Because the human brain is apt to interpret this as the intersection of the cylinder with a plane perpendicular to the axis of the cylinder, we can expect that humans will perceive $A(s)$ from E and $B(s)$ from F.

We placed the restriction that $A(s)$ and $B(s)$ be monotone in the horizontal direction and that the left and right terminal points coincide. We now ask: is it always possible to construct an ambiguous cylindrical surface if $A(s)$ and $B(s)$ satisfy the above restrictions? The answer is no. This is because the output surface of Algorithm 2 may intersect itself or one part of the edge curve may be hidden by other part of the surface.

However, it is not clear which additional conditions are needed for $A(s)$ and $B(s)$. At present, the most we can do is to confirm or reject the results, using the method described in Algorithm 3.

Algorithm 3 (post check)

Input: output surface of Algorithm 2.

Output: "accept" or "reject".

Procedure:

1. Make an orthogonal projection of the space curve $C(s)$ onto the plane T. If the resulting plane curve has self-intersections, report "reject", and stop.

2. If both of the terminal curves, i.e., $C(s)$ and its counterpart obtained by the translation of $C(s)$ along L, are partially hidden by the surface when it is seen from E, report "reject", and stop.

3. If both of the terminal curves are partially hidden by the surface when it is seen from F, report "reject" and stop.

4. Report "accept", and stop.

Thus, we can construct an ambiguous surface in the following way. First, use Algorithm 1 to construct a space curve $C(s)$, next, use Algorithm 2 to construct the cylindrical surface, and finally, use Algorithm 3 to check if we have obtained our objective. If the surface is rejected by Algorithm 3, then we can conclude that no ambiguous cylinder exists for the given pair of curves $A(s)$ and $B(s)$.

The above procedure produces an open cylinder. The garage roof in Fig. 1 was constructed in this way. For a closed cylinder such as the one in Fig. 2, we can repeat the above procedure twice, once for the upper half and once for the lower half.

For a general pair of shapes, we begin by decomposing the given pair of plane curves into horizontally monotone segments, and we then establish a one-to-one correspondence between these segments. If we succeed in doing this, we can apply the above method to each pair of horizontally monotone curves.

5 Examples

The roof of the garage in Fig. 1 is an example of an open cylinder, and Fig. 9 shows a general view of the garage. From this view, we can see that the actual shape of the garage is neither rounded nor corrugated. It still is not easy to understand the actual shape of the roof from this viewpoint (Fig. 9), and this may be because we view it filtered by our preference for rectangularity.

Fig. 9. Another view of the garage in Fig. 1.

An unfolded surface of this roof is shown in Fig. 10, and from this diagram, the reader can construct a paper model of the roof. It consists of two parts: part *A* corresponds to the roof as drawn from the back, while part *B* is a support that will maintain the correct angles in the constructed roof. The outermost lines indicate the boundaries of the parts. The inner lines are of three types: solid lines indicate mountain folds, dashed lines indicate valley folds, and dash-dot lines indicate where another part should be glued.

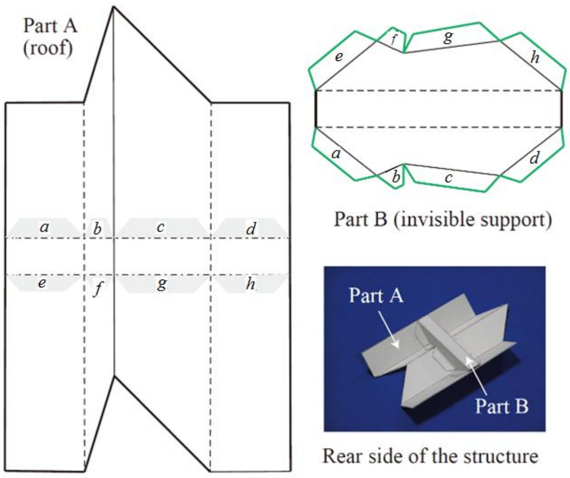


Fig. 10. Template for a paper model of the garage roof.

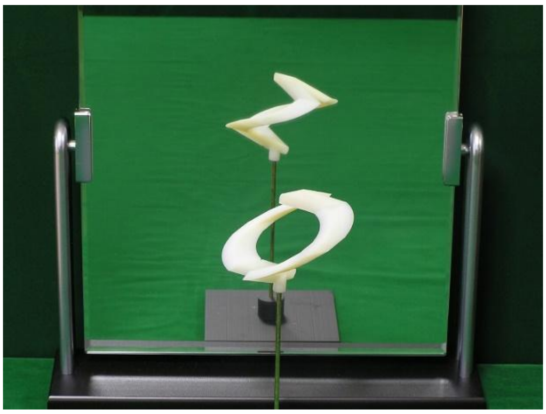
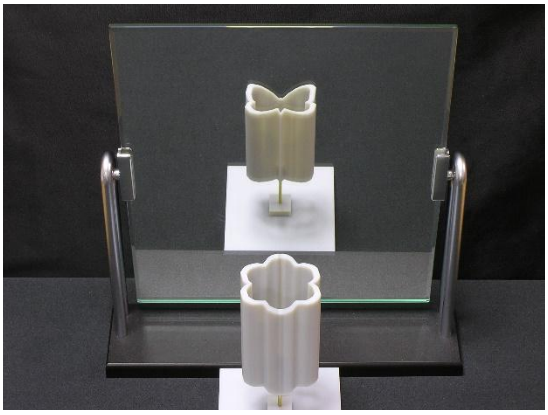


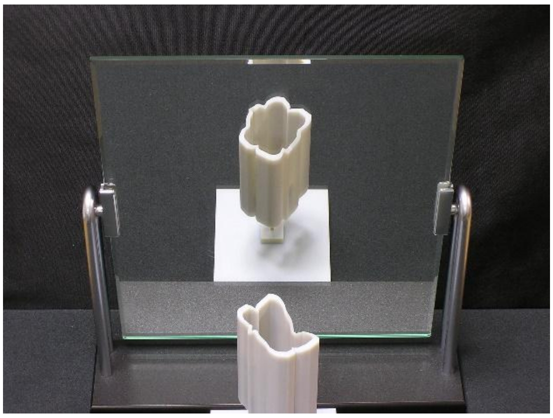
Fig. 11. Another view of the surface shown in Fig. 2.

An example of a closed cylinder that was created by a 3D printer is shown in Fig. 2. Unlike an object made from paper, this is made from a rigid material that does not require additional support. Since we do not need to hide a support, we can shorten the length of the cylinder. Another view of this surface is shown in Fig. 11, and from this view, it is clear that the true shape is neither a full moon nor a star.

Another example is shown in Fig. 12(a). Here, the surface appears to be a six-petal flower when viewed directly, but the mirror image is a butterfly. A general view of this surface is shown in Fig. 12(b). Note that the boundaries of both the star and the butterfly consist of six monotone curves; this surface was designed by applying the procedure six times, once for each of the pairs of monotone curves.



(a) special view that generates illusion



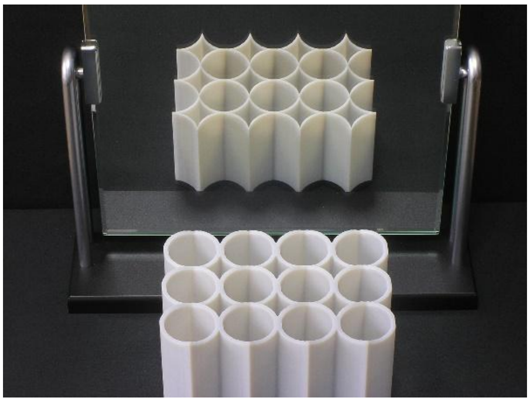
(b) general view of the same object

Fig. 12. Ambiguous cylinder “Flower and Butterfly”.

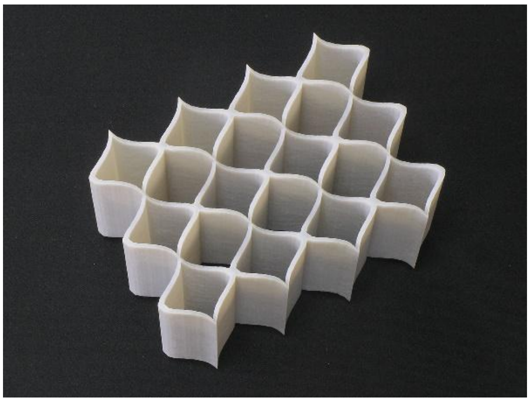
More complicated examples are shown in Figs. 13 and 14. In both figures, (a) shows the object and its mirror image seen from the special viewpoint, and (b) shows the same object seen from a general viewpoint.

6 Concluding Remarks

We have presented a new class of surfaces that drastically change appearance when viewed in a mirror, and we have described a method for designing cylindrical surfaces that create this illusion. Theoretically, the two viewpoints that generate the illusion are unique. However, our experiments show that the illusion does not disappear when the object is viewed by both eyes or even if we move our heads within a certain range. In this sense, this illusion is robust.

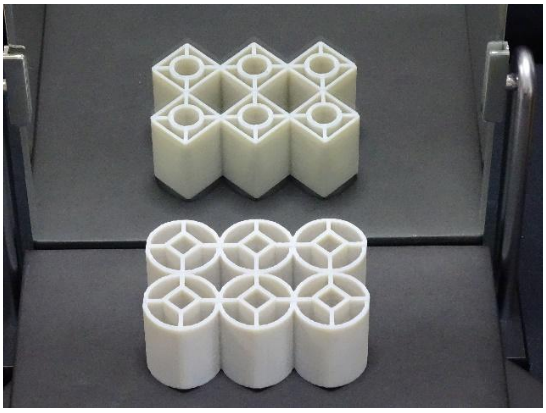


(a) cylinder and its mirror image seen from the special viewpoint

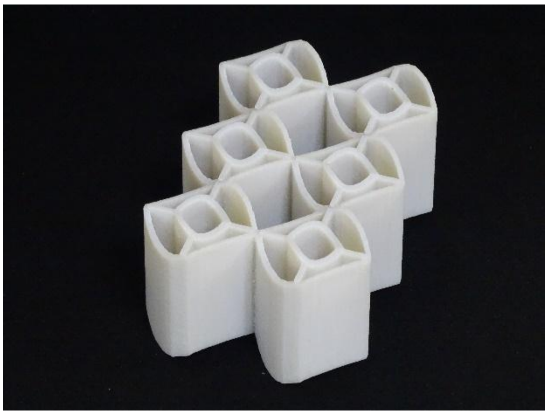


(b) the same object seen from a general viewpoint

Fig. 13. Ambiguous Cylinder "Capricious Pipes".



(a) cylinder and its mirror image seen from the special viewpoint



(b) the same object seen from a general viewpoint

Fig. 14. Ambiguous Cylinder "Circles Surround Squares or Squares Surround Circles?".

One of our future challenges is to characterize the robustness of this illusion in more detail, and to explain why it is so robust.

Another major challenge is to characterize more strictly the mathematical properties of ambiguous cylinders. In particular, at present, we can determine the existence of a particular ambiguous cylinder only after applying Algorithm 3. However, we would like to know this immediately when we are given two curves; that is, we want to find the necessary and sufficient conditions such that an ambiguous cylinder can be created for a given pair of curves $A(s)$ and $B(s)$.

Acknowledgments

(*This research is partly supported by the Grant-in-Aid for Basic Scientific Research No. 24360039, and that for Challenging Exploratory Research No. 15K12067.*)

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